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Perturbative symmetry approach

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Abstract

The aim of our paper is to formulate a perturbative version of the symmetry approach in the symbolic representation and to generalize it in order to make it suitable for the study of nonlocal and non-evolution equations. Our formalism is the development and incorporation of the perturbative approach of Zakharov and Schulman, the symbolic method of Sanders and Wang and the standard symmetry approach of Shabat *et al*. We apply our theory to describe integrable generalizations of the Benjamin–Ono type equations and to isolate integrable cases of the Camassa–Holm type equations.

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1. Introduction

In the symmetry approach the existence of infinite hierarchies of higher symmetries and/or local conservation laws is taken as a definition of integrability. The main aims of the approach are to obtain easily verifiable necessary conditions of integrability, to identify integrable cases and even to give a complete description and classification of integrable systems of a particular type. It proved to be quite successful for a description of integrable evolution equations and systems of equations [1–3]. Further progress has been achieved in [7,8] where the symbolic representation of the ring of differential polynomials enables us to reduce the problem of classification of polynomial homogeneous partial differential equations with higher symmetries to a problem of factorization of very special symmetrical polynomials and the latter has been solved by number theory methods. The aim of our paper is to formulate a perturbative version of the symmetry approach in the symbolic representation and to generalize it in order to make it suitable for the study of nonlocal and non-evolution equations. We apply our theory to describe integrable generalizations of the Benjamin–Ono type equations and to isolate integrable cases of the Camassa–Holm type equations.

2. Symmetry approach—basic definitions and facts

Here we give a brief account of basic facts, definitions and notations (for details and proofs see [1–3]). Suppose we have an evolution partial differential equation

$$u_t = F(u_n, \dots, u_1, u_0), \qquad n \geqslant 2 \tag{1}$$

where $u_0 = u(x,t), u_1 = u_x(x,t), u_2 = u_{xx}(x,t), \dots, u_n = \partial_x^n u(x,t)$. Equation (1) can be represented by two compatible infinite-dimensional dynamical systems

$$u_{0x} = u_1, u_{1x} = u_2, \dots, u_{mx} = u_{m+1}, \dots$$
$$u_{0t} = F_0, u_{1t} = F_1, \dots, u_{mt} = F_m, \dots$$

where

$$F_k(u_{n+k}, \dots, u_1, u_0) = D^k(F(u_n, \dots, u_1, u_0))$$

and the linear differential operator

$$D = u_1 \frac{\partial}{\partial u_0} + u_2 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + \cdots, \tag{2}$$

represents the derivation in x. The operator D is applied to functions of a finite number of variables and therefore only a finite number of terms in the sum (2) is required.

In the symmetry approach it is assumed that all functions such as F_k depends on a finite number of variables and belong to a proper differential field $\mathcal{F}(u, D)$ generated by u and the derivation D (2). The partial differential equation (1) defines another derivation of the field $\mathcal{F}(u, D)$:

$$D_t = F \frac{\partial}{\partial u_0} + F_1 \frac{\partial}{\partial u_1} + F_2 \frac{\partial}{\partial u_2} + \cdots, \qquad F_k \in \mathcal{F}(u, D) \tag{3}$$

commuting with D.

A symmetry of equation (1) can be defined as derivation D_τ:

$$D_\tau = G \frac{\partial}{\partial u_0} + G_1 \frac{\partial}{\partial u_1} + G_2 \frac{\partial}{\partial u_2} + \cdots, \qquad G_k \in \mathcal{F}(u, D) \tag{4}$$

of our field $\mathcal{F}(u, D)$ which commutes with derivations D and D_t. It follows from $[D, D_\tau] = 0$ that $G_k = D^k(G)$.

The Fréchet derivative of $a \in \mathcal{F}(u, D)$ is defined as a linear differential operator of the form

$$a_* = \sum_k \frac{\partial a}{\partial u_k} D^k.$$

We say that element a has order n if the corresponding differential operator a_* is of order n. The order of equation (1) is the order of F; the order of a symmetry is defined in a similar way. If a symmetry has order $n \geqslant 2$ then we call it a higher symmetry. The Lie brackets for any two elements $a, b \in \mathcal{F}(u, D)$ are defined as

$$[a, b] = a_*(b) - b_*(a).$$

In these terms the definition of the symmetry of equation (1) can be formulated as follows: function $G \in \mathcal{F}(u, D)$ generates a symmetry of equation (1) if $[F, G] = 0$.

For any $a \in \mathcal{F}(u, D)$ the time derivative $D_t(a)$ can be represented as

$$D_t(a) = a_*(F). \tag{5}$$

The variational derivative is defined as

$$\frac{\delta a}{\delta u} = \sum_k (-1)^k D^k\left(\frac{\partial a}{\partial u_k}\right).$$

Formal pseudo-differential series, which for simplicity we shall call formal series, are defined as

$$A = a_m D^m + a_{m-1}D^{m-1} + \cdots + a_0 + a_{-1}D^{-1} + a_{-2}D^{-2} + \cdots \qquad a_k \in \mathcal{F}(u, D). \tag{6}$$

The product of two formal series is defined by

$$aD^k \circ bD^m = a(bD^{m+k} + C_k^1 D(b)D^{k+m-1} + C_k^2 D^2(b)D^{k+m-2} + \cdots), \tag{7}$$

where $k, m \in \mathbb{Z}$ and the binomial coefficients are defined as

$$C_n^j = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}.$$

This product is associative.

Definition 1. *The formal series*

$$\Lambda = l_m D^m + l_{m-1}D^{m-1} + \cdots + l_0 + l_{-1}D^{-1} + l_{-2}D^{-2} + \cdots, \qquad l_k \in \mathcal{F}(u, D) \tag{8}$$

is called a formal recursion operator for equation (1) if

$$D_t(\Lambda) = F_* \circ \Lambda - \Lambda \circ F_*. \tag{9}$$

In the literature the formal recursion operator is also called the formal symmetry of equation (1).

The central result of the symmetry approach can be represented by the following theorem, which we attribute to Shabat (details of the proof and applications one can find in [1–4]):

Theorem 1. *If equation (1) has an infinite hierarchy of symmetries of arbitrary high order, then there exists a formal recursion operator.*

The theorem states that for integrable equations, i.e. equations possessing an infinite hierarchy of higher symmetries, one can solve equation (9) and determine recursively the coefficients $l_m, l_{m-1}, \ldots$ of Λ such that all these coefficients will belong to the field $\mathcal{F}(u, D)$. The solvability conditions of equation (9) can be formulated in the elegant form of a canonical sequence of local conservation laws of equation (1). They provide powerful necessary conditions of integrability. These conditions can be used for testing for integrability for a given equation or even for a complete description of integrable equations of a particular order.

3. Differential polynomials and symbolic representation

In what follows we shall consider equation (1) whose right-hand side is a differential polynomial or can be represented in the form of a series

$$F(u_n, \ldots, u_1, u_0) = F_1[u] + F_2[u] + F_3[u] + \cdots, \tag{10}$$

where $F_k[u]$ is a homogeneous differential polynomial, i.e. a polynomial of variables $u_n, \ldots, u_1, u_0$ with complex constant coefficients satisfying the condition $F_k[\lambda u] = \lambda^k F_k[u]$, $\lambda \in \mathbb{C}$, linear part $F_1[u] = L(u_0)$ and L is a linear operator ($\mathrm{ord}(L) \geqslant 2$):

$$L = \sum_{k=0}^{n} r_k D^k, \qquad r_k \in \mathbb{C}. \tag{11}$$

For such equations we develop here a perturbative method to construct a formal recursion operator and testing for integrability. For simplicity we shall consider the case when a function

F is a differential polynomial, i.e. the series (10) contains a finite number of terms. The generalization to the case of infinite series will be obvious.

Differential polynomials over $\mathbb{C}$ form a differential ring $\mathcal{R}(u, D)$ which have a natural gradation

$$\mathcal{R}(u, D) = \bigoplus_{n\geqslant 1} \mathcal{R}_n(u, D), \tag{12}$$

where $\mathcal{R}_n(u, D)$ is a set of homogeneous differential polynomials of degree n. The condition $n \geqslant 1$ in (12) means that $1 \notin \mathcal{R}(u, D)$. In order to develop a perturbation theory and for further generalization of the approach to nonlocal cases it is convenient to introduce a symbolic representation of this ring.

Symbolic representation (or symbolic method) was used in mathematics since the middle of the 19th century. It was successfully applied to the theory of integrable equations by Gelfand and Dickey [5] in 1975 and also by Zakharov and Schulman [6]. Recently the power of this method has been demonstrated again in the series of works by Sanders and Jing Ping Wang (see, for example, [7, 8]), where they have given an ultimate description of integrable hierarchies of polynomial homogeneous evolution equations.

Actually the symbolic representation is a simplified form of notations and rules for formal Fourier images of dynamical variables u_n, differential polynomials and formal series (6) with coefficients from the ring $\mathcal{R}(u, D) \oplus \mathbb{C}$.

Let $\hat{u}(\kappa, t)$ denote a Fourier image of $u(x, t)$:

$$u(x, t) = \int_{-\infty}^{\infty} \hat{u}(\kappa, t) \exp(\mathrm{i}\kappa x)\, \mathrm{d}\kappa.$$

Then we have the following correspondences: $u_0 \to \hat{u}, u_1 \to \mathrm{i}\kappa\hat{u}, \ldots, u_m \to (\mathrm{i}\kappa)^m \hat{u}, , \ldots$. The Fourier image of a monomial $u_n u_m$ can obviously be represented as

$$u_n u_m = \iiint \delta(\kappa_1 + \kappa_2 - \kappa)(\mathrm{i}\kappa_1)^n (i\kappa_2)^m \hat{u}(\kappa_1, t)\hat{u}(\kappa_2, t) \exp(\mathrm{i}\kappa x)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa \tag{13}$$

and can be rewritten in a symmetrized form:

$$u_n u_m = \iiint \delta(\kappa_1 + \kappa_2 - \kappa) \frac{[(\mathrm{i}\kappa_1)^n (\mathrm{i}\kappa_2)^m + (\mathrm{i}\kappa_2)^n (\mathrm{i}\kappa_1)^m]}{2} \times \hat{u}(\kappa_1, t)\hat{u}(\kappa_2, t) \exp(\mathrm{i}\kappa x)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa. \tag{14}$$

Therefore

$$u_n u_m \to \delta(\kappa_1 + \kappa_2 - \kappa) \frac{[(\mathrm{i}\kappa_1)^n (\mathrm{i}\kappa_2)^m + (\mathrm{i}\kappa_2)^n (\mathrm{i}\kappa_1)^m]}{2} \hat{u}(\kappa_1, t)\hat{u}(\kappa_2, t).$$

We shall simplify notations further by omitting the delta function, replacing $\mathrm{i}\kappa_n$ by ξ_n and $\hat{u}(\kappa_1, t)\hat{u}(\kappa_2, t)$ by u^2. Thus we shall represent the monomial

$u_n u_m$ by a symbol $u^2 a(\xi_1, \xi_2)$ where

$$a(\xi_1, \xi_2) = \frac{[\xi_1^n \xi_2^m + \xi_2^n \xi_1^m]}{2}$$

is a symmetrical polynomial of its arguments. Following this rule we shall represent any differential monomial $u_0^{n_0} u_1^{n_1} \cdots u_q^{n_q}$ by the symbol

$$u_0^{n_0} u_1^{n_1} \cdots u_q^{n_q} \to u^m \langle \xi_1^0 \cdots \xi_{n_0}^0 \xi_{n_0+1}^1 \cdots \xi_{n_0+n_1}^1 \xi_{n_0+n_1+1}^2 \cdots \xi_{n_0+n_1+n_2}^2 \cdots \xi_m^q \rangle$$

where $m = n_0 + n_1 + \cdots + n_q$ and the brackets $\langle\rangle$ mean the symmetrization over the group of permutation of m elements (i.e. permutation of all arguments ξ_j)

$$\langle f(\xi_1, \xi_2, \ldots, \xi_m) \rangle = \frac{1}{m!} \sum_{\sigma\in\Sigma_m} f(\sigma(\xi_1), \sigma(\xi_2), \ldots, \sigma(\xi_m)).$$

For example

$$u_n \to u\xi_1^n, \qquad u_3^2 \to u^2\xi_1^3\xi_2^3, \qquad u^3u_2 \to u^4\frac{\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2}{4}.$$

We want to emphasize that the symmetrization over the permutation group is important; it is the symmetrization that makes the symbol defined uniquely. Equality of symbols implies the equality of the corresponding differential polynomials.

The symbolic representation $\hat{\mathcal{R}}(u,\eta)$ of the differential ring $\mathcal{R}(u,D)$ can be defined as follows. The sum of differential monomials is represented by the sum of the corresponding symbols. To the multiplication of monomials f and g with symbols $f \to u^p a(\xi_1,\ldots,\xi_p)$ and $g \to u^q b(\xi_1,\ldots,\xi_q)$ corresponds the symbol

$$fg \to u^{p+q}\langle a(\xi_1,\ldots,\xi_p)b(\xi_{p+1},\ldots,\xi_{p+q})\rangle.$$

Here the symmetrization is taken over the group of permutation of all $p+q$ arguments $\xi_1,\ldots,\xi_{p+q}$. The derivative $D(f)$ of a monomial f with the symbol $u^p a(\xi_1,\ldots,\xi_s)$ is represented by

$$D(f) \to u^s(\xi_1+\xi_2+\cdots+\xi_p)a(\xi_1,\ldots,\xi_s).$$

The following rules are motivated by the theory of linear pseudo-differential operators in Fourier representation and are nothing but abbreviated notations. To the operator D (2) we shall assign a special symbol η and the following rules of action on symbols:

$$\eta(u^n a(\xi_1,\ldots,\xi_n)) = u^n a(\xi_1,\ldots,\xi_n)\sum_{j=1}^{n}\xi_j$$

and the composition rule

$$\eta\circ u^n a(\xi_1,\ldots,\xi_n) = u^n a(\xi_1,\ldots,\xi_n)\left(\sum_{j=1}^{n}\xi_j+\eta\right).$$

The latter corresponds to the Leibnitz rule $D\circ f = D(f) + fD$. Now it can be shown that the composition rule (7) can be represented as following. Let us have two operators fD^q and gD^s such that f and g have symbols $u^i a(\xi_1,\ldots,\xi_i)$ and $u^j b(\xi_1,\ldots,\xi_j)$, respectively. Then $fD^q \to u^i a(\xi_1,\ldots,\xi_i)\eta^q$, $gD^s \to u^j b(\xi_1,\ldots,\xi_j)\eta^s$ and

$$fD^q\circ gD^s \to u^{i+j}\left\langle a(\xi_1,\ldots,\xi_i)\left(\eta+\sum_{m=i+1}^{i+j}\xi_m\right)^q b(\xi_{i+1},\ldots,\xi_{i+j})\eta^s\right\rangle. \tag{15}$$

Here the symmetrization is taken over the group of permutation of all $i+j$ arguments $\xi_1,\ldots,\xi_{i+j}$: the symbol η is not included in this set. In particular, it follows from (15) that $D^q\circ D^s \to \eta^{q+s}$. The composition rule (15) is valid for positive and negative exponents q,s. In the case of positive exponents it is a polynomial in η and the result is a Fourier image of a differential operator. In the case of negative exponents one can expand the result on η at $\eta\to\infty$ in order to identify it with (7). In the symbolic representation instead of formal series (6) it is natural to consider formal series of the form

$$B = b(\eta) + ub_1(\xi_1,\eta) + u^2b_2(\xi_1,\xi_2,\eta) + u^3b_3(\xi_1,\xi_2,\xi_3,\eta)+\cdots, \qquad b(\eta)\neq 0. \tag{16}$$

Let $fD^q \to u^i a(\xi_1,\ldots,\xi_i)\eta^q$, then the symbolic representation for the formally conjugated operator is

$$(-1)^q D^q\circ f \to u^i a(\xi_1,\ldots,\xi_i)\left(-\eta-\sum_{n=1}^{i}\xi_n\right)^q.$$

The symbolic representation of the Fréchet derivative of the element $f \to u^n a(\xi_1, \dots, \xi_n)$ is

$$f_* \to nu^{n-1} a(\xi_1, \dots, \xi_{n-1}, \eta).$$

For example, let $F = u_3 + 6uu_1$, then $F \to u\xi_1^3 + 3u^2(\xi_1 + \xi_2)$ and

$$F_* \to \eta^3 + 6u(\xi_1 + \eta).$$

It is interesting to notice that the symbol of the Fréchet derivative is always symmetric with respect to all permutations of arguments, including the argument η. Moreover, the following obvious, but useful proposition holds:

Proposition 1. *A differential operator is a Fréchet derivative of an element of $\mathcal{R}(u, D)$ if and only if its symbol is invariant with respect to all permutations of its argument, including the argument η.*

The variational derivative $\delta f/\delta u$ of $f \to u^m a(\xi_1, \dots, \xi_m)$ can be represented as

$$\frac{\delta f}{\delta u} \to mu^{m-1} a\left(\xi_1, \dots, \xi_{m-1}, -\sum_{i=1}^{m-1} \xi_i\right).$$

The symbolic representation has been extended and proved to be very useful in the case of noncommutative differential rings [9]. It can be easily generalized to the case of many dependent variables [10], suitable for the study of a system of equations. Here we are going to extend it further to the case of nonlocal and multi-dimensional equations.

4. Symmetry approach in symbolic representation

Let the right-hand side of equations (1) be a differential polynomial or can be represented in the form of a series (10). In the symbolic representation it can be written as

$$u_t = u\omega(\xi_1) + \frac{u^2}{2} a_1(\xi_1, \xi_2) + \frac{u^3}{3} a_2(\xi_1, \xi_2, \xi_3) + \frac{u^4}{4} a_3(\xi_1, \xi_2, \xi_3, \xi_4) + \cdots = F, \tag{17}$$

where $\omega(\xi_1)$, $a_n(\xi_1, \dots, \xi_{n+1})$ are symmetrical polynomials and $\deg \omega(\xi_1) \geqslant 2$. According to the previous section the Fréchet derivative of the right-hand side is of the form

$$F_* = \omega(\eta) + ua_1(\xi_1, \eta) + u^2 a_2(\xi_1, \xi_2, \eta) + u^3 a_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots. \tag{18}$$

Symmetries of equation (17), if they exist, can be found recursively:

Proposition 2. *Suppose equation (17) has a symmetry*

$$u_\tau = u\Omega(\xi_1) + \sum_{j\geqslant 1} \frac{u^{j+1}}{j+1} A_j(\xi_1, \dots, \xi_{j+1}) = G. \tag{19}$$

Then functions $A_j(\xi_1, \dots, \xi_{j+1})$ of the symmetry are related to functions $a_i(\xi_1, \dots, \xi_{i+1})$ of the equation by the following formulae:

$$A_1(\xi_1, \xi_2) = \frac{N^\omega(\xi_1, \xi_2)}{N^\Omega(\xi_1, \xi_2)} a_1(\xi_1, \xi_2) \tag{20}$$

$$\begin{aligned}
A_m(\xi_1, \dots, \xi_{m+1}) = {} & \frac{N^\omega(\xi_1, \dots, \xi_{m+1})}{N^\Omega(\xi_1, \dots, \xi_{m+1})} a_m(\xi_1, \dots, \xi_{m+1}) \\
& + N^\omega(\xi_1, \dots, \xi_{m+1}) \left\langle \sum_{j=1}^{m-1} \frac{m+1}{m-j+1} A_j(\xi_1, \dots, \xi_j, \xi_{j+1} + \cdots + \xi_{m+1}) \right. \\
& \times a_{m-j}(\xi_{j+1}, \dots, \xi_{m+1}) \\
& \left. - \sum_{j=1}^{m-1} \frac{m+1}{j+1} a_{m-j}(\xi_1, \dots, \xi_{m-j}\xi_{m-j+1} + \cdots + \xi_{m+1}) A_j(\xi_{m-j+1}, \dots, \xi_{m+1}) \right\rangle
\end{aligned} \tag{21}$$

where

$$N^{\omega}(\xi_1,\ldots,\xi_m)=\left(\omega\left(\sum_{n=1}^{m}\xi_n\right)-\sum_{n=1}^{m}\omega(\xi_n)\right)^{-1},$$
$$N^{\Omega}(\xi_1,\ldots,\xi_m)=\left(\Omega\left(\sum_{n=1}^{m}\xi_n\right)-\sum_{n=1}^{m}\Omega(\xi_n)\right)^{-1}. \tag{22}$$

Proof. To find a symmetry we have to solve equation $F_*(G)=G_*(F)$ with respect to G. For $F_*(G)$ we have

$$\begin{aligned}F_*(G)=u\omega(\xi_1)\Omega(\xi_1)&+\sum_{j\geqslant1}\frac{u^{j+1}}{j+1}\omega(\xi_1+\cdots+\xi_{k+1})A_j(\xi_1,\ldots,\xi_{j+1})\\&+\sum_{i\geqslant1}\frac{u^{i+1}}{i+1}a_i(\xi_1,\ldots,\xi_{i+1})[\Omega(\xi_1)+\cdots+\Omega(\xi_{i+1})]\\&+\sum_{i\geqslant1}\sum_{j\geqslant1}\frac{u^{i+j+1}}{j+1}\langle a_i(\xi_1,\ldots,\xi_i,\xi_{i+1}+\cdots+\xi_{i+j+1})A_j(\xi_{i+1},\ldots,\xi_{i+j+1})\rangle.\end{aligned}$$

Similarly we obtain

$$\begin{aligned}G_*(F)=u\omega(\xi_1)\Omega(\xi_1)&+\sum_{i\geqslant1}\frac{u^{i+1}}{i+1}\Omega(\xi_1+\cdots+\xi_{i+1})a_i(\xi_1,\ldots,\xi_{i+1})\\&+\sum_{j\geqslant1}\frac{u^{j+1}}{j+1}A_j(\xi_1,\ldots,\xi_{j+1})[\omega(\xi_1)+\cdots+\omega(\xi_{j+1})]\\&+\sum_{i\geqslant1}\sum_{j\geqslant1}\frac{u^{i+j+1}}{i+1}\langle A_j(\xi_1,\ldots,\xi_j,\xi_{j+1}+\cdots+\xi_{i+j+1})a_i(\xi_{j+1},\ldots,\xi_{i+j+1})\rangle.\end{aligned}$$

Substituting those relations into the equation $F_*(G)=G_*(F)$ and collecting the coefficients at each power of u^j we find simple equations to determine $A_m(\xi_1,\ldots,\xi_{m+1})$. Solutions of these equations are given by (20), (21). □

For any function F of the form (17) we can solve the linear operator equation (9) and find a formal recursion operator Λ.

Proposition 3. *An operator Λ is a solution of equation (9) if its symbol is of the form*

$$\Lambda=\phi(\eta)+u\phi_1(\xi_1,\eta)+u^2\phi_2(\xi_1,\xi_2,\eta)+u^3\phi_3(\xi_1,\xi_2,\xi_3,\eta)+\cdots, \tag{23}$$

where $\phi(\eta)$ is an arbitrary function and $\phi_m(\xi_1,\ldots,\xi_m,\eta)$ are determined recursively:

$$\phi_1(\xi_1,\eta)=N^{\omega}(\xi_1,\eta)a_1(\xi_1,\eta)(\phi(\eta+\xi_1)-\phi(\eta)) \tag{24}$$

$$\begin{aligned}\phi_m(\xi_1,\ldots,\xi_m,\eta)=N^{\omega}(\xi_1,\ldots,\xi_m,\eta)\Bigg\{&(\phi(\eta+\xi_1+\cdots+\xi_m)-\phi(\eta))a_m(\xi_1,\ldots,\xi_m,\eta)\\&+\sum_{n=1}^{m-1}\Bigg\langle\frac{n}{m-n+1}\phi_n(\xi_1,\ldots,\xi_{n-1},\xi_n+\cdots+\xi_m,\eta)a_{m-n}(\xi_n,\ldots,\xi_m)\\&+\phi_n(\xi_1,\ldots,\xi_n,\eta+\xi_{n+1}+\cdots+\xi_m)a_{m-n}(\xi_{n+1},\ldots,\xi_m,\eta)\\&-a_{m-n}(\xi_{n+1},\ldots,\xi_m,\eta+\xi_1+\cdots+\xi_n)\phi_n(\xi_1,\ldots,\xi_n,\eta)\Bigg\rangle\Bigg\}.\end{aligned} \tag{25}$$

Proof. Using (5) we find

$$D_t(\Lambda) = \sum_{n\geqslant 1} u^n \phi_n(\xi_1,\ldots,\xi_n,\eta)\sum_{i=1}^{n}\omega(\xi_i) + \sum_{n\geqslant 1, m\geqslant 1}\frac{n}{m+1}u^{n+m}\langle\phi_n(\xi_1,\ldots,\xi_{n-1},\xi_n+\xi_{n+1} + \cdots + \xi_{n+m},\eta)a_m(\xi_n,\ldots,\xi_{n+m})\rangle.$$

It follows from the composition rule (15) that

$$\begin{aligned}[][F_*,\Lambda] = \sum_{m\geqslant 1} u^m &((\omega(\eta+\xi_1+\cdots+\xi_m)-\omega(\eta))\phi_m(\xi_1,\ldots,\xi_m,\eta)\\ &- (\phi(\eta+\xi_1+\cdots+\xi_m)-\phi(\eta))a_m(\xi_1,\ldots,\xi_m,\eta))\\ &+ \sum_{n\geqslant 1, m\geqslant 1} u^{n+m}\langle a_n(\xi_1,\ldots,\xi_n,\eta+\xi_{n+1}+\cdots+\xi_{n+m})\phi_m(\xi_{n+1},\ldots,\xi_{n+m},\eta)\\ &- \phi_m(\xi_1,\ldots,\xi_m,\eta+\xi_{m+1}+\cdots\xi_{m+n})a_n(\xi_{m+1},\ldots,\xi_{m+n},\eta)\rangle.\end{aligned}$$

Collecting the coefficients at each power u^m we find that the structure of the relations is triangular, and all coefficients ϕ_m can be found recursively. □

We immediately see the advantage of the perturbative approach. Now we are able to obtain explicit recursion relations for determining the coefficients of a symmetry and a formal recursion operator while in the standard symmetry approach the corresponding problem was quite difficult.

The existence of a symmetry means that all coefficients $A_m(\xi_1,\ldots,\xi_{m+1})$ are polynomials (not rational functions). In other words, the symbols $u^{m+1}A_m(\xi_1,\ldots,\xi_{m+1}) \in \hat{\mathcal{R}}(u,\eta)$ correspond to differential polynomials in the standard representation. This requirement can be used for testing for integrability and even for complete classification of integrable equations (see [7–9]).

In the standard symmetry approach the integrability, i.e. the existence of infinite hierarchies of local symmetries or conservation laws, implies (theorem 1) that all coefficients l_n are local and belong to the corresponding differential field or ring. In the symbolic representation it suggests the following definition.

Definition 2. *We say that the function $b_m(\xi_1,\ldots,\xi_m,\eta), m \geqslant 1$ is k-local if the first k coefficients $\beta_{mn}(\xi_1,\ldots,\xi_m), n = n_s,\ldots,n_s+k$ of its expansion at $\eta\to\infty$*

$$b_m(\xi_1,\ldots,\xi_m,\eta) = \sum_{n=s_n}^{\infty}\beta_{mn}(\xi_1,\ldots,\xi_m)\eta^{-n}$$

are symmetric polynomials. We say that the coefficient $b_m(\xi_1,\ldots,\xi_m,\eta)$ of a formal series (16) is local if it is k-local for any k.

Theorem 2. *Suppose equation (17) has an infinite hierarchy of symmetries:*

$$u_{t_i} = u\Omega_i(\xi_1) + \sum_{j\geqslant 1}\frac{u^{j+1}}{j+1}A_{ij}(\xi_1,\ldots,\xi_{j+1}) = G_i, \qquad i = 1,2,\ldots \tag{26}$$

where $\Omega_i(\xi_1)$ are polynomials of degree $m_i = \deg(\Omega_i(\xi_1))$ and $m_1 < m_2 < \cdots < m_i < \cdots$. Then the coefficients $\phi_m(\xi_1,\ldots,\xi_m,\eta)$ of the formal recursion operator

$$\Lambda = \eta + u\phi_1(\xi_1,\eta) + u^2\phi_2(\xi_1,\xi_2,\eta) + \cdots \tag{27}$$

are local.

Proof. We use the following obvious lemma. □

Lemma 1. *For any formal series of the form (27) let $P(\Lambda)$ be a polynomial of Λ (of degree $q \geqslant 1$) with constant coefficients. Then the first N terms of the formal series $P(\Lambda)$ are k-local if and only if the first N terms of Λ are k-local.*

In order to prove the theorem we will show by induction that for any N and k there exist q such that N first coefficients $\tilde{\phi}_1(\xi_1,\eta),\ldots,\tilde{\phi}_N(\xi_1,\ldots,\xi_N,\eta)$ of the formal recursion operator

$$\Lambda_q = \Omega_q(\Lambda) = \Omega_q(\eta) + u\tilde{\phi}_1(\xi_1,\eta) + u^2\tilde{\phi}_2(\xi_1,\xi_2,\eta) + \cdots$$

are k-local.

It follows from (20) and (22) that

$$A_{q1}(\xi_1,\xi_2) = \frac{\Omega_q(\xi_1+\xi_2) - \Omega(\xi_2)}{\omega(\xi_1+\xi_2) - \omega(\xi_1) - \omega(\xi_2)} a_1(\xi_1,\xi_2) - \frac{\Omega(\xi_1)}{\omega(\xi_1+\xi_2) - \omega(\xi_1) - \omega(\xi_2)} a_1(\xi_1,\xi_2).$$

Recalling (24) we obtain

$$\tilde{\phi}_1(\xi_1,\eta) = A_{q1}(\xi_1,\eta) + R_1(\xi_1,\eta) \tag{28}$$

where

$$R_1(\xi_1,\eta) = \frac{\Omega(\xi_1)}{\omega(\xi_1+\eta) - \omega(\xi_1) - \omega(\eta)} a_1(\xi_1,\eta).$$

Expanding the last two formulae on $1/\eta$ and taking into account that $A_{q1}(\xi_1,\eta)$ is a polynomial and therefore a local function and

$$\tilde{\phi}_1(\xi_1,\eta) = \sum_{j\leqslant s_1} \tilde{\phi}_{1j}(\xi_1)\eta^j, \qquad R_1(\xi_1,\eta) = \sum_{j\leqslant l1} r_{1j}(\xi_1)\eta^j$$

where

$$s_1 = m_q - n + n_1, \qquad l_1 = n_1 - n + 1$$

we see that at least first $s_1 - l_1 - 1 = m_q - 2$ coefficients $\tilde{\phi}_{1j}(\xi_1)$ are polynomials and therefore $\tilde{\phi}_1(\xi_1,\eta)$ is k-local and $k = m_q - 2$. Thus for any k there exists m_q such that the first coefficient $\phi_1(\xi_1,\eta)$ of Λ is k-local.

Let us assume that there exists m_q such that first $j-1$ coefficients $\phi_n(\xi_1,\ldots,\xi_n,\eta)$, $n = 1,\ldots,j-1$ are k-local. Then we will show that the coefficient $\phi_j(\xi_1,\ldots,\xi_j,\eta)$ is k-local for sufficiently large m_q.

It follows from (21) and (25) that the coefficient $\tilde{\phi}_j(\xi_1,\ldots,\xi_j,\eta)$ can be represented as

$$\tilde{\phi}_j(\xi_1,\ldots,\xi_j,\eta) = A_j(\xi_1,\ldots,\xi_j,\eta) + R_j(\xi_1,\ldots,\xi_j,\eta) \tag{29}$$

where

$$\begin{aligned} R_j(\xi_1,\ldots,\xi_j,\eta) = {} & -N^{\omega}(\xi_1,\ldots,\xi_j,\eta) \\ & \times \Bigg[\sum_{i=1}^{j-1} \Bigg(\langle R_i(\xi_1,\ldots,\xi_i,\xi_{i+1}+\cdots+\xi_j+\eta) a_{j-i}(\xi_{i+1},\ldots,\xi_j,\eta)\rangle \\ & - \langle a_{j-i}(\xi_1,\ldots,\xi_{j-i},\xi_{j-i+1}+\cdots+\xi_j+\eta) R_i(\xi_{i+1},\ldots,\xi_j,\eta)\rangle \\ & + \frac{i}{j-i+1} \langle R_i(\xi_1,\ldots,\xi_{i-1},\xi_i+\cdots+\xi_j,\eta) a_{j-i}(\xi_i,\ldots,\xi_j,)\rangle \\ & - \frac{j-i}{i+1} \langle a_{j-i}(\xi_1,\ldots,\xi_{j-i-1},\xi_{j-i}+\cdots+\xi_j,\eta) A_i(\xi_{j-i},\ldots,\xi_j)\rangle \Bigg) \\ & - (\Omega_q(\xi_1)+\cdots+\Omega(\xi_j)) a_j(\xi_1,\ldots,\xi_j,\eta) \Bigg]. \end{aligned}$$

Taking the expansion of (29) in $1/\eta$ we find that the principal power of η for $\tilde{\phi}_j$ can be represented as $m_q + S(n, n_1, \ldots, n_j)$ and the principal power for R_j as $Q(n, n_1, \ldots, n_j)$. Therefore for sufficiently large m_q we will have $m_q + S(n, n_1, \ldots, n_j) - Q(n, n_1, \ldots, n_j) > k$, i.e. the coefficient ϕ_j is k-local.

For any N and k there exists such m_q that first N coefficients of Λ are k-local. □

The symmetry approach in symbolic representation suggests the following test for integrability of equations of the form (17):

- Find a first few coefficients $\phi_n(\xi_1, \ldots, \xi_n, \eta)$ (first three nontrivial coefficients ϕ_n were sufficient to analyse in all cases known to us).
- Expand these coefficients in series of $1/\eta$:

$$\phi_n(\xi_1, \ldots, \xi_n, \eta) = \sum_{s=s_n} \Phi_{ns}(\xi_1, \ldots, \xi_n)\eta^{-s} \tag{30}$$

  and find the corresponding functions $\Phi_{ns}(\xi_1, \ldots, \xi_n)$.
- Check that functions $\Phi_{ns}(\xi_1, \ldots, \xi_n)$ are polynomials (not rational functions).

As an example of an application we consider equations of the form

$$u_t = u\omega(\xi_1) + \sum_{i\geqslant 1} \frac{u^{i+1}}{i+1} a_i(\xi_1, \ldots, \xi_{i+1}), \tag{31}$$

where $\omega(\xi_1)$ is a polynomial on ξ_1 of the degree $\deg(\omega(\xi_1)) = n \geqslant 2$ and $a_i(\xi_1, \ldots, \xi_{i+1})$ are symmetric polynomials on its arguments of degree $\deg(a_i(\xi_1, \ldots, \xi_{i+1})) \leqslant n - 2$. The following propositions are valid.

Proposition 4. *If equation (31) is integrable, then $\omega(0) = 0$, i.e. the polynomial $\omega(\xi_1)$ can be factorized $\omega(\xi_1) = \xi_1 f(\xi_1)$, where $f(\xi_1)$ is a polynomial.*

Proposition 5. *Suppose n is even and $a_1(\xi_1, \xi_2) \equiv 0$ or n is odd and $a_1(\xi_1, \xi_2) \equiv 0$, $a_2(\xi_1, \xi_2, \xi_3) \equiv 0$. Then equation (31) is not integrable.*

Proof of proposition 5. Let us consider first the case of even n. Let we have the equation of the form (31), where $a_1(\xi_1, \xi_2) = \cdots = a_{s-1}(\xi_1, \ldots, \xi_s) = 0$, $a_s(\xi_1, \ldots, \xi_{s+1}) \neq 0$. For a corresponding Λ-operator using (24) and (25) we have

$$\Lambda = \eta + \sum_{j\geqslant s} u^j \phi_j(\xi_1, \ldots, \xi_j, \eta).$$

Let us consider the coefficient

$$\phi_s(\xi_1, \ldots, \xi_s, \eta) = \frac{(\xi_1 + \cdots + \xi_s)a_s(\xi_1, \ldots, \xi_s, \eta)}{\omega(\xi_1 + \cdots + \xi_s + \eta) - \omega(\eta) - \omega(\xi_1) - \cdots - \omega(\xi_s)}.$$

For $s \geqslant 2$ the polynomial $\omega(\xi_1 + \cdots + \xi_s + \eta) - \omega(\eta) - \omega(\xi_1) - \cdots - \omega(\xi_s)$ is of degree n and cannot be factorized [7]. Therefore it cannot divide the numerator, which is a factorizable polynomial of degree $n - 1$.

This coefficient is nonlocal. Indeed, the expansion of denominator $\omega(\xi_1 + \cdots + \xi_s + \eta) - \omega(\eta) - \omega(\xi_1) - \cdots - \omega(\xi_s)$ at $\eta \to \infty$ is of the form

$$[\omega(\xi_1 + \cdots + \xi_s + \eta) - \omega(\eta) - \omega(\xi_1) - \cdots - \omega(\xi_s)]^{-1}$$
$$= \frac{1}{\omega'(\eta)(\xi_1 + \cdots + \xi_s)} \sum_{j\geqslant 0} \left[\frac{\omega(\xi_1) + \cdots + \omega(\xi_s)}{\omega'(\eta)(\xi_1 + \cdots + \xi_s)} - \sum_{i=2}^{n} \frac{\omega^{(i)}(\eta)}{\omega'(\eta)} \frac{(\xi_1 + \cdots + \xi_s)^{i-1}}{i!} \right]^j$$

and it contains powers of a singular term $\frac{\omega(\xi_1)+\cdots+\omega(\xi_s)}{(\xi_1+\cdots+\xi_s)}$ (for $s \geqslant 1$ and even n the sum $\omega(\xi_1) + \cdots + \omega(\xi_s) \not\equiv 0$ on the hyperplane $\xi_1 + \cdots + \xi_s = 0$.

In the case of odd n the proof is analogous, with the only difference being that the function $\omega(\xi_1+\cdots+\xi_s+\eta)-\omega(\eta)-\omega(\xi_1)-\cdots-\omega(\xi_s)$ is not factorizable if $s \geqslant 3$ and $\omega(\xi_1)+\omega(-\xi_1)$ may equal zero for some dispersion laws $\omega(k)$. The proposition is proved. □

Proof of proposition 4. We consider a polynomial dispersion law $\omega(k)$ and we need to proof that $\omega(0) = 0$. Let the degree of the $\omega(k)$ n be odd. As follows from proposition 4, if equation (31) is integrable then either $a_1(\xi_1,\xi_2) \neq 0$ or $a_1(\xi_1,\xi_2) = 0$, but $a_2(\xi_1,\xi_2,\xi_3) \neq 0$. Let us consider the coefficient $\phi_1(\xi_1,\eta)$ of the corresponding Λ-operator in the case $a_1(\xi_1,\xi_2) \neq 0$:

$$\phi_1(\xi_1,\eta) = \frac{\xi_1 a_1(\xi_1,\eta)}{\omega(\xi_1+\eta)-\omega(\xi_1)-\omega(\eta)}.$$

Its expansion on η at the point $\eta \to \infty$ is of the form

$$\phi_1(\xi_1,\eta) = \frac{a_1(\xi_1,\eta)}{\omega'(\eta)} \sum_{j\geqslant 0} \left[\frac{\omega(\xi_1)}{\xi_1\omega'(\eta)} - \sum_{i=2}^{n} \omega^{(i)}(\eta)\frac{\xi_1^i}{i!}\right]^j$$

and contains a singularity $\omega(\xi_1)/\xi_1$ in any power. If $\omega(0) \neq 0$ then the polynomial $\omega(\xi_1+\eta) - \omega(\xi_1) - \omega(\eta)$ is not factorizable. It cannot divide the numerator $\xi_1 a_1(\xi_1,\eta)$ since $\deg a_1(\xi_1,\eta) < \deg \omega(\eta)$.

The cases of even n and $a_1(\xi_1,\xi_2) = 0$, $a_2(\xi_1,\xi_2,\xi_3) \neq 0$, odd n can be proved in a similar way. □

The statement of proposition 5 in the homogeneous case was proved in the work of Sanders and Wang [8].

5. Nonlocal extensions

The main goal of this paper is to extend the symmetry approach to the case of nonlocal and non-evolutionary equations. Multi-dimensional integrable equations and their hierarchies also have intrinsic nonlocality and certain modifications of the standard symmetry approach are required [11]. The symbolic representation seems to be suitable to tackle the problem of nonlocality.

Here we consider two types of nonlocal equations. The first one is the Benjamin–Ono equation:

$$u_t = H(u_2) + 2uu_1 \tag{32}$$

where $H(f)$ denotes the Hilbert transform:

$$H(f) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{y-x}\,\mathrm{d}y. \tag{33}$$

The second example is

$$m_t = cmu_1 + um_1, \qquad m = u - u_2, \qquad c \neq 0, \tag{34}$$

where c is a constant (this equation is known to be integrable for $c = 2$ [12] and 3 [13], see also [14]). Our approach can be extended to the multi-dimensional case, where nonlocality is an intrinsic property (we are going to present the multi-dimensional case in a separate paper).

Equation (32) is nonlocal (not differential) and system (34) is non-evolutionary. Their higher symmetries, when they exist, are even more nonlocal. To tackle the problem of nonlocality a proper extension of the differential ring is required. In the (2+1)-dimensional case such an extension was proposed in [11]. Here we are developing this idea and reformulating it in the frame of a perturbative symmetry approach and we illustrate it by examples.

5.1. Benjamin–Ono type equations

The Benjamin–Ono equation (32) contains the Hilbert transform. It is well known that its higher symmetries and conservation laws contain nested Hilbert transforms and we have to extend the differential ring $\mathcal{R}(u, D)$ in order to study structures associated with the Benjamin–Ono type equations.

The construction of the extension is quite similar to [11]. Let us consider the following sequence of ring extensions:

$$\mathcal{R}_H^0 = \mathcal{R}(u, D), \qquad \mathcal{R}_H^1 = \overline{\mathcal{R}_H^0 \bigcup H(\mathcal{R}_H^0)}, \qquad \mathcal{R}_H^{n+1} = \overline{\mathcal{R}_H^n \bigcup H(\mathcal{R}_H^n)}, \tag{35}$$

where the set $H(\mathcal{R}_H^n)$ is defined as $H(\mathcal{R}_H^n) = \{H(a); a \in \mathcal{R}_H^n\}$ and the horizontal line denotes the ring closure. Each $\mathcal{R}_H^n$ is a ring and the upper index n indicates the nesting depth of the operator H.

Elements of $\mathcal{R}_H^0$ are differential polynomials. Elements of $\mathcal{R}_H^n n \geqslant 1$ we call *quasi-local functions*. The right-hand side of equation (32), its symmetries and densities of conservation laws are quasi-local functions.

In the symbolic representation operator H is represented by $\mathrm{i}\,\mathrm{sign}(\eta)$. Now the symbolic representation of the ring extensions is obvious. Suppose element $A \in \mathcal{R}_H^0$ and the corresponding symbol is $u^n a(\xi_1, \ldots, \xi_n)$, then $H(A)$ is represented by the symbol $u^n b(\xi_1, \ldots, \xi_n)$, where $b(\xi_1, \ldots, \xi_n) = \mathrm{i}\,\mathrm{sign}(\xi_1 + \cdots + \xi_n) a(\xi_1, \ldots, \xi_n)$.

In symbolic representation all definitions, such as the Fréchet derivative and formal recursion operator, are exactly the same as in the local case. Now we have more freedom in the choice of function $\phi(\eta)$ (proposition 3). We can choose $\phi(\eta) = \eta$ or $\phi(\eta) = \eta\,\mathrm{sign}(\eta)$. The asymptotic locality conditions, i.e. the conditions that the coefficients $u^n \Phi_{nm}(\xi_1, \ldots, \xi_n)$ of the formal recursion operator Λ (23) represent symbols of differential polynomials (and therefore $\Phi(\xi_1, \ldots, \xi_n)$ are symmetrical polynomials) now should be replaced by the quasi-locality conditions. Namely, that the coefficients $u^n \Phi_{nm}(\xi_1, \ldots, \xi_n)$ represent symbols of quasi-local functions means that the coefficients $u^n \Phi_{nm}(\xi_1, \ldots, \xi_n)$ correspond to elements from the extended ring.

For example, let us consider Benjamin–Ono equation (32). In symbolic representation it follows that

$$u_t = \mathrm{i}u\,\mathrm{sign}(\xi_1)\xi_1^2 + u^2(\xi_1 + \xi_2). \tag{36}$$

The first coefficient $\phi_1(\xi_1, \eta)$ of the corresponding formal recursion operator $\Lambda = \eta + u\phi_1(\xi_1, \eta) + u^2\phi_2(\xi_1, \xi_2, \eta) + \cdots$ looks like

$$\phi_1(\xi_1, \eta) = \mathrm{sign}(\eta) + \frac{\xi_1(\mathrm{sign}(\xi_1) + \mathrm{sign}(\eta))}{2\eta} + \mathrm{O}\left(\frac{1}{\eta^7}\right)$$

and it is evident that it is quasi-local. One may easily check quasi-locality of the other coefficients $\phi_2(\xi_1, \xi_2, \eta), \phi_3(\xi_1, \xi_2, \xi_3, \eta), \ldots$.

As another illustration let us consider an equation of the form

$$u_t = \hat{H}(u_2) + c_1 u u_1 + c_2 \hat{H}(u u_1) + c_3 u \hat{H}(u_1) + c_4 u_1 \hat{H}(u) + c_5 \hat{H}(u \hat{H}(u_1)) + c_6 \hat{H}(u)\hat{H}(u_1) \tag{37}$$

where c_j are complex constants. Where $\hat{H} = -\mathrm{i}H$ so the symbol for $\hat{H}$ is now sign). We announce here the following theorem (for details and proof see [15])

Theorem 3. *An equation of the form (37) is integrable if and only if it is, up to the point transformation $u \to au + b\hat{H}(u)$, $a^2 - b^2 \neq 0$, one of the list*

$$u_t = \hat{H}(u_2) + D(\tfrac{1}{2}c_1u^2 + c_2u\hat{H}(u) + \tfrac{1}{2}c_1\hat{H}(u)^2) \qquad (38)$$

$$u_t = \hat{H}(u_2) + D(\tfrac{1}{2}c_1u^2 + \tfrac{1}{2}c_2\hat{H}(u^2) - c_2u\hat{H}(u)) \qquad (39)$$

$$u_t = \hat{H}(u_2) + uu_1 \pm \hat{H}(uu_1) \mp u\hat{H}(u_1) \mp 2u_1\hat{H}(u) + \hat{H}(u\hat{H}(u_1)) \qquad (40)$$

$$u_t = \hat{H}(u_2) + \hat{H}(uu_1) + u_1\hat{H}(u) \pm \hat{H}(u\hat{H}(u_1)) \pm \hat{H}(u)\hat{H}(u_1). \qquad (41)$$

As the last example in this subsection let us consider an equation of the form

$$u_t = u\omega(\xi_1) + \frac{u^2}{2}(\xi_1 + \xi_2). \qquad (42)$$

Proposition 6. *Let the dispersion relation $\omega(k)$ for equation (42) be of the form $\omega(k) = k^2 f(k)$ where the function $f(k) \to 1$ is faster than any power of k when $k \to \infty$ and $f(k) \to c_{-1}k^{-1} + c_0 + c_1k + c_2k^2 + \cdots$ when $k \to 0$. Then equation (42) is integrable if and only if $f(k) = 1$ or $f(k) = \coth(\frac{k}{c_1})$.*

In the standard (x) representation equation (42) has the form

$$u_t = \frac{1}{2\pi}\int \xi_1^2 f(\xi_1)u(\xi_1)\mathrm{e}^{\mathrm{i}\xi_1 x}\,\mathrm{d}\xi_1 + uu_1 = T(u_2) + uu_1.$$

The function $f(k) = 1$ corresponds to the Burgers equation, while $f(k) = \coth(\frac{k}{c_1})$ corresponds to the gravity waves on the finite depth water (see, for example, [16]). In the last case $f(k)$ can be made nonsingular at $k = 0$ by the transformation $f(k) \to f(k) - c_1/k$, which corresponds to the Galilean transformation $\omega(k) \to \omega(k) - c_1k$. The Benjamin–Ono equation corresponds to the limit $c_1 \to 0$.

Proof. Quasi-local extension by pseudo-differential operator T can be done in a similar way as with the Hilbert operator $\hat{H}$ (35). The coefficient $\phi_1(\xi_1, \eta)$ of the corresponding formal recursion operator $\Lambda = \eta + u\phi_1(\xi_1, \eta) + u^2\phi_2(\xi_1, \xi_2, \eta) + u^3\phi_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots$ is quasi-local. Indeed,

$$\phi_1(\xi_1, \eta) = \frac{\xi_1 a(\xi_1, \eta)}{\omega(\xi_1 + \eta) - \omega(\xi_1) - \omega(\eta)} = \frac{a(\xi_1, \eta)}{2\eta\left(1 + \frac{\xi_1 - \xi_1 f(\xi_1)}{2\eta}\right)}, \qquad \eta \to \infty.$$

Let us consider the expansion on $1/\eta$ of the next coefficient $\phi_2(\xi_1, \xi_2, \eta)$:

$$\phi_2(\xi_1, \xi_2, \eta) = \phi_{22}(\xi_1, \xi_2)\eta^{-2} + \phi_{23}(\xi_1, \xi_2)\eta^{-3} + \phi_{24}(\xi_1, \xi_2)\eta^{-4} + \cdots,$$

where functions $\phi_{22}(\xi_1, \xi_2)$ and $\phi_{23}(\xi_1, \xi_2)$ are quasi-local (we do not present here explicit expressions; they are rather large), while the function $\phi_{24}(\xi_1, \xi_2)$ can be represented in the form

$$\phi_{24}(\xi_1, \xi_2) = \frac{F(\xi_1, \xi_2)}{\xi_1 + \xi_2},$$

where $F(\xi_1, \xi_2)$ is a quasi-local function. In general, the function $\phi_{24}(\xi_1, \xi_2)$ does not correspond to any element of our extended differential ring and is the obstacle to the integrability. This obstacle would be absent if $F(\xi_1, \xi_2)$ were divisible by $\xi_1 + \xi_2$. The conditions of divisibility are

$$c_1(f(-\xi_1) + f(\xi_1)) = 0 \qquad (43)$$

and

$$(f(-\xi_1) + f(\xi_1))(f(-\xi_1)f(\xi_1) - c_0f(-\xi_1) - c_0f(\xi_1) + 1) = 0.$$

If $f(k)$ is regular at zero and is not an odd function $c_1 = 0$ then from the last equation it is easy to see that $f(k) \equiv 1$. If $f(-k) = -f(k)$ then it can be shown that the coefficient $\phi_2(\xi_1, \xi_2, \eta)$ is quasi-local. Making the expansion of the next coefficient $\phi_3(\xi_1, \xi_2, \xi_3, \eta)$ we obtain

$$\phi_3(\xi_1, \xi_2, \xi_3, \eta) = \phi_{33}(\xi_1, \xi_2, \xi_3)\eta^{-3} + \phi_{34}(\xi_1, \xi_2, \xi_3)\eta^{-4} + \cdots,$$

where the function $\phi_{33}(\xi_1, \xi_2, \xi_3)$ is quasi-local while $\phi_{34}(\xi_1, \xi_2, \xi_3)$ can be represented in the form

$$\phi_{34}(\xi_1, \xi_2, \xi_3) = \frac{G(\xi_1, \xi_2, \xi_3)}{\xi_1 + \xi_2 + \xi_3},$$

where $G(\xi_1, \xi_2, \xi_3)$ is a quasi-local function. The coefficient $\phi_{34}(\xi_1, \xi_2, \xi_3)$ is quasi-local if $G(\xi_1, \xi_2, \xi_3)$ is divisible by $\xi_1 + \xi_2 + \xi_3$ and the divisibility condition yields the following differential equation:

$$-6g + 6zg' + z^2 g'' = 0, \qquad g = f^2 + c_1 f' - 1,$$

whose general solution is

$$g = \alpha z + \frac{\beta}{z^6}.$$

Taking into account the Laurent series at zero we have $\alpha = \beta = 0$ and for (43) we find

$$f = \coth\left(\frac{z}{c_1}\right). \qquad \square$$

5.2. *Camassa–Holm–Degasperis equation*

Equation (34) can be obviously rewritten as a scalar evolution nonlocal equation

$$u_t = \Delta(-uu_3 + (c+1)uu_1 - cu_1u_2), \qquad c \neq 0, \tag{44}$$

where the operator $\Delta = (1 - D^2)^{-1}$. It is also well known that its higher symmetries (when they exist) contain a nested operator Δ and we need to extend the differential ring $\mathcal{R}(u, D)$ in a similar way as in the previous section:

$$\mathcal{R}^0_\Delta = \mathcal{R}(u, D), \qquad \mathcal{R}^1_\Delta = \overline{\mathcal{R}^0_\Delta \bigcup \Delta(\mathcal{R}^0_\Delta)}, \qquad \mathcal{R}^{n+1}_\Delta = \overline{\mathcal{R}^n_\Delta \bigcup \Delta(\mathcal{R}^n_\Delta)}. \tag{45}$$

A symbolic representation of the operator Δ is $\Delta \to \frac{1}{1-\eta^2}$. The symbolic representation of elements of differential rings $\mathcal{R}^n_\Delta$ is obvious. For example, if A is an element from $\mathcal{R}^0_\Delta$ with corresponding symbol $u^n a(\xi_1, \ldots, \xi_n)$ then $\Delta(A)$ has a symbol $u^n \frac{a(\xi_1,\ldots,\xi_n)}{1-(\xi_1+\cdots+\xi_n)^2}$.

The procedure for testing the integrability of a given equation is exactly the same as in the previous case. Let us apply it to the Camassa–Holm–Degasperis type equations.

Theorem 4. *Equation (44) is integrable only if $c = 2$ or 3*

Proof. In order to introduce a linear term in (44) we shift $u \to u - 1$

$$u_t = (1 - D^2)^{-1}(u_3 - (c+1)u_1 - uu_3 + (c+1)uu_1 - cu_1u_2).$$

In the symbolic representation it can be rewritten as follows:

$$u_t = u\omega(\xi_1) + \frac{u^2}{2}a(\xi_1, \xi_2) = F$$

where

$$\omega(k) = \frac{k^3 - (c+1)k}{1 - k^2}$$

$$a(\xi_1, \xi_2) = \frac{(c+1)(\xi_1 + \xi_2) - (\xi_1^3 + \xi_2^3) - c\xi_1\xi_2(\xi_1 + \xi_2)}{1 - (\xi_1 + \xi_2)^2}.$$

Calculating the first two coefficients of the corresponding formal recursion operator

$$\Lambda = \eta + u\phi_1(\xi_1, \eta) + u^2\phi_2(\xi_1, \xi_2, \eta) + \cdots.$$

The first coefficient

$$\phi_1(\xi_1, \eta) = \frac{(\xi_1^2 - 1)(\eta^2 - 1)(\xi_1^2 + \eta^2 - \xi_1\eta - 1 + c(\xi_1\eta - 1))}{c\eta(\eta^2 + \xi_1^2 + \xi_1\eta - 3)}$$

is quasi-local because coefficients of its expansion in $1/\eta$ are polynomials on ξ_1. For the second coefficient $\phi_2(\xi_1, \xi_2, \eta)$ we have the following expansion:

$$\phi_2(\xi_1, \xi_2, \eta) = \Phi_{21}(\xi_1, \xi_2)\eta + \Phi_{20}(\xi_1, \xi_2) + \Phi_{2,-1}(\xi_1, \xi_2)\eta^{-1} + \Phi_{2,-2}(\xi_1, \xi_2)\eta^{-2} + \Phi_{2,-3}(\xi_1, \xi_2)\eta^{-3} + \cdots,$$

where coefficients $\Phi_{21}(\xi_1, \xi_2), \ldots, \Phi_{2,-2}(\xi_1, \xi_2)$ are polynomials on its arguments (we do not present here explicit expressions for this function—they are quite large), while the coefficient $\Phi_{2,-3}$ has the form

$$\Phi_{2,-3}(\xi_1, \xi_2) = \frac{f(\xi_1, \xi_2)}{1 - \xi_1\xi_2}$$

and $f(\xi_1, \xi_2)$ is polynomial. If $f(\xi_1, \xi_2)$ does not have $1 - \xi_1\xi_2$ as a factor, then the symbol $u^2\Phi_{2,-3}$ does not correspond to any element of our extended ring and hence is the obstacle to integrability for equation (44). Polynomial $f(\xi_1, \xi_2)$ can be divided by $1 - \xi_1\xi_2$ only if the condition

$$(c - 2)(c - 3) = 0$$

is satisfied and in these cases the coefficient $\Phi_{2,-3}(\xi_1, \xi_2)$ is a polynomial. It is well known that equation (44) is integrable if $c = 2, 3$ (see [12–14]). □

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